

# MATH3210 - SPRING 2022 - SECTION 001

## EXAM 1 - REVIEW SHEET

### HOW TO USE THIS REVIEW SHEET

On the exam or problems, you may use any of the definitions and theorems stated on the review sheet, *unless you are explicitly asked to prove a theorem listed here*. Any unnamed theorem you may use without citing. If you use a named theorem, cite that theorem by name when invoking its conclusions. Please don't hesitate to ask questions. I've proofread this, but typos may still lurk!

#### 1. THE REAL NUMBERS, SUPREMA AND INFIMA

We didn't discuss many of the definitions in the following, but the main conclusions of the following are that the real numbers are well-defined and have suprema and infima for their subsets. Crucially, it also tells us that all of the basic arithmetic manipulations hold for both equalities and inequalities (as long as we pay close attention to signs when working with inequalities!).

**Theorem 1.**  $\mathbb{R}$  is a complete ordered field, and can be constructed from  $\mathbb{Q}$  by Dedekind cuts.<sup>1</sup>

**Definition 1.** If  $A \subset \mathbb{R}$ , let

$$\sup A = \begin{cases} \text{lub}(A), & A \text{ is bounded above} \\ \infty, & \text{otherwise} \end{cases} \quad / \quad \inf A = \begin{cases} \text{glb}(A), & A \text{ is bounded below} \\ -\infty, & \text{otherwise} \end{cases}$$

*Remark 1.* The key difference between sup and lub is that sup always exists as the least upper bound of a set when it exists, and  $\infty$  when it doesn't, but at the cost that we use the *extended* real numbers to define it.

**Theorem 2** (Nested Interval Property). For each  $n \in \mathbb{N}$ , let  $I_n = [a_n, b_n]$  be a closed interval of real numbers with endpoints  $a_n < b_n$ . If the intervals  $\{I_n\}$  are nested (ie,  $I_{n+1} \subset I_n$ ), then  $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ .

#### 2. SEQUENCES OF REAL NUMBERS

**Definition 2.** If  $\{a_n\}$  is a sequence of real numbers, we say that  $\{a_n\}$  converges to  $a$ , or that  $a$  is limit of the sequence if for every  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that if  $n \geq N$ ,  $|a_n - a| < \varepsilon$ . We write  $a_n \rightarrow a$  or  $\lim_{n \rightarrow \infty} a_n = a$  as shorthand for “ $\{a_n\}$  converges to  $a$ .”

**Theorem 3** (Limit rate theorem). Let  $\{a_n\}$  be a sequence of real numbers, and  $\{b_n\}$  be a sequence of real numbers converging to 0. If for every  $n \in \mathbb{N}$ ,  $|a_n - a| < b_n$  then  $\{a_n\}$  converges to  $a$ .

**Theorem 4** (Limit arithmetic theorem). Let  $\{a_n\}$  and  $\{b_n\}$  be sequences of real numbers, and  $c \in \mathbb{R}$ . Then:

- (a) If  $a_n \rightarrow a$ , then  $ca_n \rightarrow ca$ .
- (b) If  $a_n \rightarrow a$  and  $b_n \rightarrow b$ , then  $a_n + b_n \rightarrow a + b$ .
- (c) If  $a_n \rightarrow a$  and  $b_n \rightarrow b$ , then  $a_n b_n \rightarrow ab$ .
- (d) If  $a_n \rightarrow a$ ,  $b_n \rightarrow b$ ,  $b \neq 0$  and  $b_n \neq 0$  for all  $n \in \mathbb{N}$ , then  $a_n/b_n \rightarrow a/b$ .

<sup>1</sup>If you'd like to dig deeper, look up any unknown terms in the following statement, which gives  $\mathbb{R}$  the “uniqueness” property for fields like this: If  $F$  is a complete ordered field, there exists a unique order-preserving field isomorphism  $\phi : F \rightarrow \mathbb{R}$ .

(e) If  $k \in \mathbb{N}$ ,  $a_n \rightarrow a$ , then  $a_n^k \rightarrow a^k$ .

(f) If  $k \in \mathbb{N}$ ,  $a_n \rightarrow a$ , then  $a_n^{1/k} \rightarrow a^{1/k}$ .

**Theorem 5.** Let  $\{a_n\}$  be a sequence of real numbers. Then  $a_n \rightarrow a$  if and only if for every  $\varepsilon > 0$ ,  $|a_n - a| \geq \varepsilon$  for only finitely many  $n \in \mathbb{N}$ .

**Theorem 6.** If  $\{a_n\}$  is a convergent sequence of real numbers, then  $\{a_n\}$  is bounded. (Equivalently, if  $\{a_n\}$  is an unbounded sequence, then it diverges)

**Definition 3.** If  $\{a_n\}$  is a sequence of real numbers, we say that  $\{a_n\}$  diverges to  $\infty$ , or that  $\lim_{n \rightarrow \infty} a_n = \infty$ , if for every  $B \in \mathbb{R}$ , there exists  $N \in \mathbb{N}$  such that if  $n \geq N$ ,  $a_n > B$ .

*Remark 2.* It is important to note the difference between the statements “ $\{a_n\}$  is unbounded” and “ $\{a_n\}$  diverges to  $\infty$ .” See the practice problems.

**Theorem 7** (Monotone Convergence Theorem). Let  $\{a_n\}$  be a increasing or decreasing sequence of real numbers. Then  $\{a_n\}$  has a limit, and that limit is finite if and only if  $\{a_n\}$  is bounded above (below).

**Theorem 8** (Limit Comparison Theorem). Let  $a_n \rightarrow a$  and  $b_n \rightarrow b$ . Assume that for every  $n \in \mathbb{N}$ ,  $a_n \leq b_n$ . Then  $a \leq b$ .

**Theorem 9** (Limit Squeeze Theorem). Let  $\{a_n\}$ ,  $\{b_n\}$  and  $\{c_n\}$  be sequences of real numbers, and assume that  $a_n \leq b_n \leq c_n$  for every  $n \in \mathbb{N}$ , that  $a_n \rightarrow x$ , and that  $c_n \rightarrow x$ . Then  $b_n \rightarrow x$ .

### 3. SUBSEQUENCES AND CAUCHY SEQUENCES

**Definition 4.** If  $\{a_n\}$  is a sequence, a *subsequence* is another sequence  $\{a_{n_k}\}$  which is indexed by  $k$ . It is determined by a strictly increasing sequence of natural numbers  $\{n_k\}$ .

**Theorem 10.** If  $a_n \rightarrow a$ , then  $a_{n_k} \rightarrow a$  for every subsequence of  $\{a_n\}$ .

**Theorem 11** (Bolzano-Weierstrass). Let  $\{a_n\}$  be a bounded sequence of real numbers. Then  $\{a_n\}$  has a convergent subsequence.

**Definition 5.** If  $\{a_n\}$  is a sequence of real numbers, we say that  $\{a_n\}$  is *Cauchy* if for every  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that if  $m, n \geq N$ ,  $|a_m - a_n| < \varepsilon$ .

**Theorem 12** (Topological Completeness Theorem). A sequence of real numbers  $\{a_n\}$  converges if and only if it is Cauchy.

### 4. LIMIT SUPERIORS AND LIMIT INFERIORS

**Definition 6.** We make the definition for  $\limsup$ . The definition for  $\liminf$  is parallel. Fix a sequence of real numbers  $a_n$ . Given  $m \in \mathbb{N}$ , let  $u_m = \sup\{a_n : n \geq m\}$ . Then define:

$$\limsup_{n \rightarrow \infty} a_n = \lim_{m \rightarrow \infty} u_m$$

**Theorem 13.**  $\limsup$  and  $\liminf$  are well-defined (ie, the sequence  $u_m$  either converges, or diverges to  $\infty$ ). Furthermore, if  $\{a_{n_k}\}$  is any subsequence of  $\{a_n\}$ , then:

$$\liminf_{n \rightarrow \infty} a_n \leq \lim_{k \rightarrow \infty} a_{n_k} \leq \limsup_{n \rightarrow \infty} a_n.$$

**Theorem 14** ( $\liminf/\limsup$  convergence criterion). If  $\{a_n\}$  is a sequence, then  $a_n \rightarrow a$  if and only if  $\liminf a_n = \limsup a_n = a$ . Furthermore,  $\{a_n\}$  diverges to  $\infty$  if and only if  $\liminf a_n = \infty$ .

## 5. PRACTICE EXERCISES

**Problem 1.** Decide whether the sequence converges or diverges. Prove that your answer is correct. If it diverges, determine if it diverges to  $\infty$ , or just diverges. If it converges, try proving that it converges with the limit arithmetic theorem, and then again without it.

- (a)  $\left\{ \frac{2n+3}{8-5n} \right\}$   
 (b)  $\left\{ \frac{n^2}{n+2} \right\}$   
 (c)  $\left\{ \frac{1}{n+\sin(n)} \right\}$   
 (d)  $\left\{ \frac{n}{2^n} \right\}$  [*Hint:* Consider the sequence  $(\frac{2}{3})^n$  and apply the Limit rate theorem]  
 (e)  $\left\{ \frac{2^n+n}{2^n+1} \right\}$   
 (f) Let  $b_n = \frac{1}{2}((-1)^n + 1)$ . Find an explicit piecewise formula for  $\{b_n\}$  and determine if it converges.  
 (g)  $\left\{ \frac{1}{b_n + 1/n} \right\}$ , where  $b_n$  is as in the previous problem.

**Problem 2.** Fix a sequence  $\{a_n\}$ . Consider the following possible definitions for increasing:

- (Stepwise) For every  $n \in \mathbb{N}$ ,  $a_{n+1} \geq a_n$ .
- (Totally) If  $m > n$  are natural numbers, then  $a_m \geq a_n$ .

Show that  $\{a_n\}$  is stepwise increasing if and only if it is totally increasing. (This implies that the two conditions are equivalent, so we just call such a sequence *increasing*.)

**Problem 3.** Explain the difference between “ $\{a_n\}$  diverges to  $\infty$ ” and “ $\{a_n\}$  is unbounded.”

**Problem 4.** Consider the definition of  $\limsup$  (Definition 6). Write the definition of  $\liminf$ .

**Problem 5.** Prove or find a counterexample: a subsequence  $\{a_{n_k}\}$  converges to  $a$  if and only if for every  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that if  $n_k \geq N$ ,  $|a_{n_k} - a| < \varepsilon$ .

*Remark 3.* At first glance, Problem 5 appears to be the definition of convergence for  $\{a_{n_k}\}$ , but there is a subtle difference. Since  $\{a_{n_k}\}$  is indexed by  $k$  and not  $n$ , the threshold for convergence should be based on the choice of  $k$ , not  $n_k$ . Make sure you understand this difference before you approach this problem! The proof may seem to run in circles if you don't correctly understand this feature.

**Problem 6.** Prove or find a counterexample: If  $\{a_n\}$  is a sequence of real numbers and  $\liminf a_n \leq x \leq \limsup a_n$ , then there exists a subsequence  $\{a_{n_k}\}$  such that  $a_{n_k} \rightarrow x$ .

**Problem 7.** Find the  $\liminf$  and  $\limsup$  of the sequences in Problem 1.

**Problem 8.** Read Definition 3, then make compatible definition for sequences diverging to  $-\infty$ . Formulate and prove a version of (the second half of) Theorem 14 for sequences diverging to  $-\infty$ .